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Preferred pattern of convection in a porous layer with a spatially non-uniform boundary temperature

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The problem of finite-amplitude thermal convection in a porous layer between two horizontal walls with different mean temperatures is considered when spatially non-uniform temperature with amplitude L^* is prescribed at the lower wall. The nonlinear problem of three-dimensional convection for values of the Rayleigh number close to the classical critical value is solved by using a perturbation technique. Two cases are considered: the wavelength $\gamma_n^{(b)}$ of the nth mode of the modulation is equal to or not equal to the critical wavelength γ_c for the onset of classical convection. The preferred mode of convection is determined by a stability analysis in which arbitrary infinitesimal disturbances are superimposed on the steady solutions. The most surprising results for the case $\gamma_n^{(b)} = \gamma_c$ for all n are that regular or non-regular solutions in the form of multi-modal pattern convection can become preferred in some range of L^*, provided the wave vectors of such pattern are contained in the set of wave vectors representing the spatially non-uniform boundary temperature. There can be critical value(s) L_c^* of L^* below which the preferred flow pattern is different from the one for $L^* > L_c^*$. The most surprising result for the case $\gamma_n^{(b)} \neq \gamma_c$ and $\gamma_n^{(b)} \equiv \gamma^{(b)}$ for all n is that some three-dimensional solution in the form of multi-modal convection can be preferred, even if the boundary modulation is one-dimensional, provided that the wavelength of the modulation is not too small. Here $\gamma^{(b)}$ is a constant independent of n.

1. Introduction

The classical problem of thermal convection in a horizontal and symmetric porous layer with prescribed temperatures at the boundaries has been the subject of investigation by many authors in the past. The linear stability for the onset of convective flow was first investigated theoretically by Lapwood (1948). Notable subsequent nonlinear investigations of the problem were due to Palm, Weber & Kvernvold (1972), Straus (1974) and Jospeh (1976). These and other investigations of this so-called perfect problem established, in particular, the following results. The linear problem is self-adjoint. The conduction state is unique for Rayleigh number R below $R_c = 4\pi^2$. Here $R = \beta g k \Delta T d \rho_0 c/(\nu\lambda)$, where β is the coefficient of thermal expansion, g is the acceleration due to gravity, k is the Darcy permeability coefficient, ΔT is the temperature difference across the layer, d is the depth of the layer, ρ_0 is the reference fluid density, c is the specific heat at constant pressure, λ is the thermal conductivity of the porous medium (fluid–solid mixture), ν is the kinematic viscosity and R_c is the critical value of R below which there is no motion. The first bifurcation, which takes place at $R = R_c$, is thus supercritical, and the two-dimensional rolls with the critical wavenumber $\alpha_c = \pi = 2\pi/\gamma_c$ are the preferred mode of convection.

As was pointed out in Riahi (1983) and confirmed in Riahi (1985), the problem of thermal convection in a porous medium is simpler than the corresponding one in an ordinary medium, but the main qualitative features of thermal convection in these two systems are the same at least for the critical regime where $R \approx R_c$ and for large-Prandtl-number fluid. Hence the problem of thermal convection in a porous medium can conveniently be used to study nonlinear effects such as the preferred flow pattern.

This paper studies the problem of the preferred pattern of convection at small amplitude in a horizontal porous layer with a spatially non-uniform temperature prescribed at the lower boundary.

This problem is an example of an imperfect bifurcation driven by imperfect heating and/or cooling, and there are a number of relevant studies in the literature. Keller (1966) studied differential heating effects in a model of thermal convection. Reiss (1976) applied the method of matched asymptotic expansions to a number of imperfect bifurcation problems. Matkowski & Reiss (1977) presented an asymptotic theory to analyse perturbations of bifurcations of the solutions of nonlinear problems. Kelly & Pal (1978) and Pal & Kelly (1978) investigated two-dimensional thermal convection with one-dimensional spatially periodic boundary conditions. Tavantzis, Reiss & Matkowsky (1978) applied their theory of singular perturbations in a mathematical and systematic manner to the case of a bounded layer with a rather arbitrary one-dimensional variable temperature imposed on one boundary. Hall & Walton (1978) considered the case of a bounded fluid layer with constant temperature on the horizontal boundaries but with non-adiabatic endwalls. Erneux & Cohen (1983) examined imperfect bifurcation near a double eigenvalue. Walton (1982, 1983) investigated the onset of thermal convection in a fluid layer of either slowly increasing depth or when the temperature difference between the horizontal boundaries is a monotonic function of a single horizontal variable. More recently, Krettenauer & Schumann (1989, 1992) carried out direct numerical simulation of Rayleigh–Bénard convection for the case where the lower surface height varied sinusoidally in one direction and for both laminar and turbulent flow regimes. Some discussion of their results will be given in §5.

The present investigation is aimed at examining the effects of a spatially non-uniform boundary condition upon the pattern of convection. It turns out that the following two general types of variation in the boundary condition lead to the same qualitative results for the present case where the convective flow is considered in a range close to the onset of convection based on classical theory (Lapwood 1948): (i) a bounding surface which is plane but its temperature varies with respect to the horizontal variables x and y along the surface; (ii) a surface which has constant temperature but is corrugated, so that the gap size between the two walls varies with respect to x and y.

The general problem under consideration can have practical values in that one might want to roughen a boundary to enhance the transport process or to control the flow structure. This later practical aspect of the problem is the main motivation for the present study which is concerned with the preferred convection pattern(s).

Kelly & Pal (1978) investigated the problem of two-dimensional convection with one-dimensional spatially periodic boundary conditions. They assumed that the amplitude of the spatial non-uniformities is small, and they set the wavelength equal to the critical wavelength for the onset of Rayleigh–Bénard convection. They determined the Nusselt number as a function of Rayleigh number R, Prandtl number, and modulation amplitude. Their assumption of two-dimensionality of the problem, however, posed a severe restriction on the analysis, and, in particular, pattern selection mechanisms could not be studied. Pal & Kelly (1978) considered the onset of two-dimensional thermal convection when the one-dimensional variations of the temperatures of the walls are in the special form of a sine wave. Here the wavelength of the modulation is assumed to be different from the critical wavelength. They found that the modulation can be stabilizing. The present investigation extends the work of both Kelly & Pal (1978) and Pal & Kelly (1978) to arbitrary three-dimensional flows and an arbitrary one- or two-dimensional non-uniform temperature boundary condition at the lower wall. We have found a number of interesting results. In particular, we found for the first time that a non-regular flow pattern can be preferred in some range of modulation amplitude even if the spatially non-uniform boundary temperature represents a regular pattern. Here by a 'regular flow pattern' we mean a pattern of the horizontal structure of the flow solution whose wave vectors $\boldsymbol{k}_n (n = -N, \ldots, -1, 1, \ldots, N$; N is a positive integer) all have the same magnitude and where the angles w between the two consecutive wave vectors all have the same value (Busse 1967). Examples of regular patterns are those due to two-dimensional rolls ($N = 1$, $w = 180°$), square cells ($N = 2$, $w = 90°$) and hexagonal cells ($N = 3$, $w = 60°$). A solution other than a regular one is called a non-regular solution. Examples of non-regular patterns are those due to rectangular cells ($N = 2$, $w = w_1^\circ$ and $180° - w_1^\circ$, where $0 < w_1^\circ < 180°$ and $w_1^\circ \neq 90°$) and six-sided polygonal cells ($N = 3$, $w = w_2^\circ$, w_3° and $180° - w_2^\circ - w_3^\circ$, where $0 < w_2^\circ + w_3^\circ < 180°$ with $w_2^\circ = w_3^\circ = 60°$ discarded). The non-regular solutions include the so-called semi-regular solutions where each wave vector encloses the angle $2\pi/N$ with each of its second nearest neighbours on either side. An example of a semi-regular pattern is the rectangular pattern defined above.

Let us now designate the amplitudes of convection and the non-uniform boundary temperature by ϵ and $L^* = \delta L$, respectively, where it is assumed that $\epsilon \ll 1$, $\delta \ll 1$ and L is an order-one quantity. Kelly & Pal (1978) investigated the two-dimensional non-trivial resonant wavelength excitation case where $L^* = 0(\epsilon^3)$. They demonstrated (Kelly & Pal 1976, 1978) that this case corresponds to the range $R \approx R_c$ ($R - R_c = 0(\epsilon^2)$). This result can be derived from their modal equation of the form

$$\epsilon^3 = \epsilon[(R-R_c)/R_c c_2] + \delta c_1, \tag{1.1}$$

where c_1 and c_2 are real constants. See Kelly & Pal (1976, 1978) for further details. The modal equation (1.1) represents the scalings for a steady-state Landau-type equation which includes the effect of the boundary modulation. The same relation between δ and ϵ was obtained by Tavantzis *et al.* (1978). For non-resonant wavelength excitation, Pal & Kelly (1978) applied a double series expansion in powers of δ and ϵ for each of the dependent variables and for R. In the present three-dimensional problem, we use the same procedures as Kelly & Pal (1978) and Pal & Kelly (1978).

2. Mathematical formulation

We consider an infinite horizontal porous layer of average depth d filled with fluid and heated from below. The layer is bounded above and below by two plane surfaces whose mean temperatures are $\bar{T}_u$ and $\bar{T}_l$, respectively. We choose to scale the temperature T^* on the basis of $\Delta T = \bar{T}_l - \bar{T}_u$. It is convenient to introduce a Cartesian system of coordinates, with the origin on the centreplane of the layer and with the z-coordinate in the vertical direction (opposite to the direction of the gravity force). We shall examine the effects of lower boundary modulations at a fixed value of ΔT and represent the magnitude of such variation relative to ΔT by δ. We define a temperature relative to the conduction state by

$$T^*(x,y,z,t) = (\tfrac{1}{2}d - z)\frac{\Delta T}{d} + T(x,y,z,t). \tag{2.1}$$

It is convenient to use non-dimensional variables in which lengths, velocities, time and temperature T are scaled respectively by d, $\lambda/d\rho_0 c$, $d^2\rho_0 c/\lambda$ and $\Delta T/R$. Here λ is the thermal conductivity of the porous medium (fluid–solid mixture), ρ_0 is the reference density of the fluid, c is the specific heat at constant pressure and R is the Rayleigh number (defined in §1). Then, with the usual Boussinesq approximation that density variations are taken into account only in the buoyancy term, the Darcy–Boussinesq–Oberbeck equations in the limit of infinite Prandtl–Darcy number (Joseph 1976) are

$$0 = -\boldsymbol{\nabla} P + \theta \boldsymbol{z} - \boldsymbol{u}, \tag{2.2a}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, \tag{2.2b}$$

$$\frac{\partial\theta}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\theta = R\boldsymbol{u}\cdot\boldsymbol{z} + \nabla^2\theta. \tag{2.2c}$$

Here θ is the dimensionless T, $\boldsymbol{u}$ is the velocity vector, P is the modified deviation of pressure from its static value and $\boldsymbol{z}$ is a unit vector in the vertical direction.

The velocity vector $\boldsymbol{u}$ in (2.2) is defined according to Darcy's law as an average over the microscale of the porous medium. We shall assume that the microscale is small enough compared with any scale size of the flow for $\boldsymbol{u}$ to remain a well-defined quantity.

The physically appropriate infinite value of the Prandtl–Darcy number follows from the extraordinarily small values of the permeability coefficient in most porous materials.

The boundary conditions for $\boldsymbol{u}$ and θ are

$$\boldsymbol{u}\cdot\boldsymbol{z} = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2}, \tag{2.3a}$$

$$\theta = \delta R h(x,y) \quad \text{at} \quad z = -\tfrac{1}{2}, \tag{2.3b}$$

$$\theta = 0 \quad \text{at} \quad z = \tfrac{1}{2}, \tag{2.3c}$$

where $h(x,y)$ is a given spatially non-uniform function of x and y. Since we have used the Darcy assumption in order to replace $\nabla^2\boldsymbol{u}$ with $-\boldsymbol{u}$ in (2.2), we cannot impose boundary conditions on the tangential components of $\boldsymbol{u}$.

The governing equations (2.2) can be simplified by using the representation

$$\boldsymbol{u} = \boldsymbol{\Omega}\phi, \quad \boldsymbol{\Omega} \equiv \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{z} \tag{2.4}$$

for the divergence-free velocity vector field $\boldsymbol{u}$ (Riahi 1983). Taking the vertical component of the curl of the curl of (2.2a) and using (2.4) in (2.2c) yields

$$\Delta_2(\nabla^2\phi + \theta) = 0, \tag{2.5a}$$

$$\left(\nabla^2 - \frac{\partial}{\partial t}\right)\theta - R\Delta_2\phi = \boldsymbol{\Omega}\phi\cdot\boldsymbol{\nabla}\theta, \tag{2.5b}$$

where $$\Delta_2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$$

Equations (2.5) must then be solved subject to the boundary conditions (2.3*b*), (2.3*c*) and

$$\phi = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2}. \tag{2.6}$$

An alternative lower boundary condition is that of a constant-temperature corrugated rigid boundary, where the conditions at the lower boundary for both ϕ and θ are prescribed at $z = -\frac{1}{2}+\delta h$. It turns out that the qualitative results of the present formulation based on such lower boundary conditions do not differ from the corresponding results based on the lower boundary conditions given by (2.3*b*) and (2.6). (See Appendix A).

3. The case of resonant wavelength excitation ($\gamma_n^{(b)} = \gamma_c$)

This case corresponds to the critical regime where $R \approx R_c$ and $L^* = O(\epsilon^3)$ (Kelly & Pal 1978). We consider the following expansions for ϕ, θ and R in powers of ϵ:

$$\begin{pmatrix}\phi\\ \theta\\ R\end{pmatrix} = \begin{pmatrix}0\\ 0\\ R_0\end{pmatrix} + \epsilon\begin{pmatrix}\phi_1\\ \theta_1\\ R_1\end{pmatrix} + \epsilon^2\begin{pmatrix}\phi_2\\ \theta_2\\ R_2\end{pmatrix} + \dots \tag{3.1}$$

and set $\delta = \epsilon^3$. Upon inserting (3.1) into (2.3*b*), (2.3*c*), and (2.5)–(2.6) and disregarding the quadratic terms, we find the linear problem

$$\Delta_2(\nabla^2\phi_1+\theta_1) = 0, \tag{3.2a}$$

$$\nabla^2\theta_1 - R_0\Delta_2\phi_1 = 0, \tag{3.2b}$$

$$\phi_1 = \theta_1 = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2}. \tag{3.2c}$$

This system is the classical linear system (Riahi 1983). The general solution of (3.2) can be written as

$$\begin{pmatrix}\phi_1\\ \theta_1\end{pmatrix} = \begin{pmatrix}f(z)\\ g(z)\end{pmatrix} W(x,y), \tag{3.3}$$

where the function W has the representation

$$W(x,y) = \sum_{n=-N}^{N} C_n W_n \equiv \sum_{n=-N}^{N} C_n \exp(\mathrm{i}\boldsymbol{k}_n\cdot\boldsymbol{r}), \tag{3.4}$$

and satisfies the relation

$$\Delta_2 W = -\alpha^2 W, \quad \langle WW\rangle = 1. \tag{3.5}$$

Here the angle brackets indicate an average over the fluid layer, $\boldsymbol{r}$ is the horizontal position vector, $\mathrm{i} = \sqrt{(-1)}$, α is the horizontal wavenumber of the flow structure, N is a positive integer, and the horizontal wavenumber vectors $\boldsymbol{k}_n$ of the flow structure satisfy the properties

$$\boldsymbol{k}_n\cdot\boldsymbol{z} = 0, \quad |\boldsymbol{k}_n| = \alpha, \quad \boldsymbol{k}_{-n} = -\boldsymbol{k}_n. \tag{3.6}$$

The coefficients C_n satisfy the conditions

$$\sum_{n=-N}^{N} C_n C_n^* = 1, \quad C_n^* = C_{-n}, \tag{3.7}$$

where the asterisk indicates the complex conjugate.

Following Riahi (1983), we have the following results:

$$\left.\begin{aligned} f(z) = \sqrt{2}\cos\pi z, \quad g(z) = (\pi^2+\alpha^2)\sqrt{2}\cos\pi z, \\ R_0 = \alpha^{-1}(\pi^2+\alpha^2)^2, \quad R_c = 4\pi^2, \quad \alpha_c = \pi. \end{aligned}\right\} \tag{3.8}$$

At the order ϵ^2, (2.3*b*), (2.3*c*) and (2.5)–(2.6) become

$$\Delta_2(\nabla^2\phi_2+\theta_2) = 0, \tag{3.9a}$$

$$\nabla^2\theta_2 - R_0\Delta_2\phi_2 - R_1\Delta_2\phi_1 = \boldsymbol{\Omega}\phi_1\cdot\boldsymbol{\nabla}\theta_1, \tag{3.9b}$$

$$\phi_2 = \theta_2 = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2}. \tag{3.9c}$$

The system (3.9) is of the classical type (Riahi 1983).

Since the classical problem is self-adjoint, the solvability conditions for the equations of higher order in ϵ require us to define the following special solutions ϕ_{1n} and θ_{1n} of the linear system of equations (3.2):

$$\begin{pmatrix}\phi_{1n}\\ \theta_{1n}\end{pmatrix} = \begin{pmatrix} f(z)\\ g(z)\end{pmatrix} W_n. \tag{3.10}$$

Multiplying (3.9*a*) by ϕ_{1n}^*, (3.9*b*) by $-R_0^{-1}\theta_{1n}^*$, adding and averaging over the whole layer and using (3.9*c*) yields $R_1 = 0$ (Riahi 1983). Equations (3.9) then yield

$$\phi_2 = \sum_{l,p=-N,\, l\neq -p}^{l,p=N} F(z,\hat{\phi}_{lp})C_l C_p W_l W_p + G(z), \tag{3.11c}$$

$$\theta_2 = -\mathrm{D}^2G(z) - \sum_{l,p=-N,\, l\neq -p}^{l,p=N} [\mathrm{D}^2 - 2\alpha^2(1+\hat{\phi}_{lp})]F(z,\hat{\phi}_{lp})C_l C_p W_l W_p, \tag{3.11b}$$

where

$$\hat{\phi}_{lp} = \alpha^{-2}(\boldsymbol{k}_l\cdot\boldsymbol{k}_p),$$

and $\mathrm{D} = \mathrm{d}/\mathrm{d}z$. The expressions for the functions F and G are given by (Riahi 1983)

$$F(z,\hat{\phi}_{lp}) = \frac{\pi(1-\hat{\phi}_{lp})\sin 2\pi z}{2[4+2(1+\hat{\phi}_{lp})+(1+\hat{\phi}_{lp})^2]}, \tag{3.12a}$$

$$G(z) = \tfrac{1}{4}\pi\sin 2\pi z. \tag{3.12b}$$

At the order ϵ^3, (2.3*b*), (2.3*c*) and (2.5)–(2.6) become

$$\Delta_2(\nabla^2\phi_3+\theta_3) = 0, \tag{3.13a}$$

$$\nabla^2\theta_3 - R_0\Delta_2\phi_3 - R_2\Delta_2\phi_1 = \boldsymbol{\Omega}\phi_1\cdot\boldsymbol{\nabla}\theta_2 + \boldsymbol{\Omega}\phi_2\cdot\boldsymbol{\nabla}\theta_1, \tag{3.13b}$$

$$\phi_3 = \theta_3 - R_0 h = 0 \quad \text{at} \quad z = -\tfrac{1}{2}, \tag{3.13c}$$

$$\phi_3 = \theta_3 = 0 \quad \text{at} \quad z = \tfrac{1}{2}. \tag{3.13d}$$

The function h given in (3.13*c*) is assumed to have the following arbitrary representation:

$$h(x,y) = R_c^{-1}\sum_{n=-N^{(b)}}^{N^{(b)}} LC_n^{(b)}W_n^{(b)} \equiv R_c^{-1}\sum_{h=-N^{(b)}}^{N^{(b)}} LC_n^{(b)}\exp(\mathrm{i}\boldsymbol{k}_n^{(b)}\cdot\boldsymbol{r}), \tag{3.14}$$

where L is a constant, $N^{(b)}$ is a positive integer which may tend to infinity and the horizontal wavenumber vectors $\boldsymbol{k}_n^{(b)}$ satisfy the properties

$$\boldsymbol{k}_n^{(b)}\cdot\boldsymbol{z} = 0, \quad |\boldsymbol{k}_n^{(b)}| = \alpha_n^{(b)} \equiv 2\pi/\gamma_n^{(b)}, \quad \boldsymbol{k}_{-n}^{(b)} = -\boldsymbol{k}_n^{(b)}. \tag{3.15}$$

The coefficients $C_n^{(b)}$ satisfy the condition

$$\sum_{n=-N^{(b)}}^{N^{(b)}} C_n^{(b)} C_n^{*(b)} = 1, \quad C_n^{*(b)} = C_{-n}^{(b)}. \tag{3.16}$$

We shall assume that $\alpha_n^{(b)} = \alpha_c = \pi$ (Riahi 1983). In general, $\alpha_n^{(b)}$ are not all the same as α_c for different n. Most of these latter cases will be considered in §4, while the rest, mentioned in §5 will be discussed in detail in a future contribution.

Multiplying (3.13a) by ϕ_{1n}^*, (3.13a) by $-R_0^{-1}\,\theta_{1n}^*$, adding and averaging over the whole layer and using (3.13c) yields

$$R_2 F_0 C_n = -2(\surd 2)\,\pi^3 L \sum_{m=-N^{(b)}}^{N^{(b)}} C_m^{(b)} \langle W_n^* W_m^{(b)}\rangle + \sum_{l \neq -p} [-(\hat{\phi}_{ml}+\hat{\phi}_{mp}) F_1 + F_2]\, C_m C_l C_p \langle W_n^* W_m W_l W_p\rangle + G_1 C_n$$

$$(n = -N, \ldots, -1, 1, \ldots, N), \tag{3.17}$$

where F_1 and F_2 are functions of $\hat{\phi}_{lp}$ and are given by

$$F_1 = -\alpha^2 \langle g\mathrm{D}f[\mathrm{D}^2-\alpha_s^2]F\rangle, \tag{3.18a}$$

$$F_2 = -\alpha^2 \langle fg[\mathrm{D}^2-\alpha_s^2]\,\mathrm{D}F\rangle, \tag{3.18b}$$

$$G_1 = -\alpha^2\langle fg\mathrm{D}^3 G\rangle, \quad F_0 = \alpha^2\langle fg\rangle, \quad \alpha_s = \alpha[2(1+\hat{\phi}_{lp})]^{\frac{1}{2}}. \tag{3.18c}$$

For $L = 0$, the expression for R_2 given by (3.17) is the same as the corresponding one for the classical problem (Riahi 1983). Hence, (3.17) can be written in the following form:

$$(R_2 - R_{2c}) F_0 C_n = -2(\surd 2)\,\pi^3 L \sum_{m=-N^{(b)}}^{N^{(b)}} C_m^{(b)} \langle W_n^* W_m^{(b)}\rangle \quad (n = -N, \ldots, -1, 1, \ldots, N), \tag{3.19}$$

where R_{2c} denotes the classical expression for R_2 (Riahi 1983).

To distinguish the physically realizable solution(s) among all possible steady solutions, the stability of ϕ, θ with respect to arbitrary three-dimensional disturbances $\tilde{\phi}, \tilde{\theta}$ is investigated. The equations and the boundary conditions for the time-dependent disturbances with addition of a time dependence of the form $\exp(\sigma t)$ are given by

$$\Delta_2(\nabla^2\tilde{\phi} + \tilde{\theta}) = 0, \tag{3.20a}$$

$$-\sigma\tilde{\theta} + \nabla^2\tilde{\theta} - R\Delta_2\,\tilde{\phi} = \boldsymbol{\Omega}\tilde{\phi}\cdot\nabla\theta + \boldsymbol{\Omega}\phi\cdot\nabla\tilde{\theta}, \tag{3.20b}$$

$$\tilde{\phi} = \tilde{\theta} = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2}. \tag{3.20c}$$

When (3.1) is used in (3.20) it becomes evident that the system (3.20) can be solved by an expansion of the form

$$\begin{pmatrix} \tilde{\phi} \\ \tilde{\theta} \\ \sigma \end{pmatrix} = \begin{pmatrix} \tilde{\phi}_1 \\ \tilde{\theta}_1 \\ \sigma_0 \end{pmatrix} + \epsilon \begin{pmatrix} \tilde{\phi}_2 \\ \tilde{\theta}_2 \\ \sigma_1 \end{pmatrix} + \ldots. \tag{3.21}$$

The solutions to the stability problem can be obtained in direct analogy to that discussed by Riahi (1983) for the classical problem ($\delta = 0$ case). The solvability conditions in the orders ϵ^n $(n = 0, 1, 2)$ lead to the following results for the most critical disturbances:

$$\sigma_0 = \sigma_1 = 0, \tag{3.22a}$$

$$(\sigma_2 - \sigma_{2c})\langle g^2\rangle C_n = -2(\sqrt{2})\pi^3 L \sum_{m=-N^{(b)}}^{N^{(b)}} C_m^{(b)} \langle W_n^* W_m^{(b)}\rangle, \tag{3.22b}$$

where σ_{2c} denotes the classical expression for σ_2 given by Riahi (1983).

Using (3.19), (3.22) and the results given by Riahi (1983) for R_{2c} and σ_{2c}, we obtain the following non-trivial results. For sufficiently large $L, R_2 < 0$, and thus the corresponding solution is subcritical $(R < R_c)$. For sufficiently large L, there may be more than one stable solution. However, the preferred solution corresponds to the one for which R is a minimum.

Let us now consider the following few specific examples in order to illustrate the non-trivial and often surprising inter-relations between the boundary modulation pattern and the subsequence preferred flow pattern:

Example 1. $N^{(b)} = 1$, $C_m^{(b)} = 1/\sqrt{2}$ and $\alpha_m^{(b)} = \alpha_c$ for all m. The terms on the right-hand sides of (3.19) and (3.22) are non-zero only if at least one of the wave vectors $\boldsymbol{k}_n$ is in the direction of $\boldsymbol{k}_m^{(b)}$. If none of the wave vectors $\boldsymbol{k}_n$ are along $\boldsymbol{k}_m^{(b)}$ for all m, then the right-hand-side terms in (3.19) and (3.22) are zero, $R_2 = R_{2c}$, $\sigma_2 = \sigma_{2c}$ and the preferred flow pattern is in the form of two-dimensional rolls (Riahi 1983) with $R_2 = \pi^4$. Orientational degeneracy of the solutions, however, is not removed by the boundary modulation effect. If one of the wave vectors $\boldsymbol{k}_n$ is along, say, $\boldsymbol{k}_1^{(b)}$ then we find from Riahi (1983), (3.19) and (3.22) that two-dimensional rolls are preferred and that

$$R_2 = \pi^4 - 2(\sqrt{2})\pi^3 L, \tag{3.23a}$$

$$\sigma_2 < \sigma_{2c}. \tag{3.23b}$$

The expression for σ_{2c} is negative only for two-dimensional roll convection. Any other solutions (three-dimensional solutions) are not allowed by the nonlinear system, unless $L = 0$. We find from (3.23a) that subcritical instability occurs for $L > \pi/2(\sqrt{2})$. The results discussed above indicate that all the two-dimensional roll solutions parallel to any direction are stable. However, rolls parallel to $\boldsymbol{k}_1^{(b)}$ have smaller R, as evidenced from (3.23), and are, therefore, preferred.

Example 2. $N^{(b)} = 2$, $C_m^{(b)} = \frac{1}{2}$ and $\alpha_m^{(b)} = \alpha_c$ for all m. The terms on the right-hand sides of (3.19) and (3.22b) are non-zero only if at least one $\boldsymbol{k}_n$ is along $\boldsymbol{k}_m^{(b)}$. If no $\boldsymbol{k}_n$ are along $\boldsymbol{k}_m^{(b)}$, then the right-hand sides of (3.19) and (3.22b) are zero, $R_2 = R_{2c} = \pi^4$, $\sigma_2 = \sigma_{2c}$ and the two-dimensional rolls with oriental degeneracy are stable. If one of $\boldsymbol{k}_n$ is along $\boldsymbol{k}_m^{(b)}$, then we find from Riahi (1983), (3.19) and (3.22) that two-dimensional roll solutions are stable, satisfying (3.23b) and

$$R_2 = \pi^4 - 2\pi^3 L. \tag{3.24}$$

The expression for σ_{2c} is negative only for two-dimensional roll convection. These stable rolls can be either along $\boldsymbol{k}_1^{(b)}$ (or $\boldsymbol{k}_{-1}^{(b)}$) or along $\boldsymbol{k}_2^{(b)}$ (or $\boldsymbol{k}_{-2}^{(b)}$), and they satisfy (3.24). For three-dimensional convection in the form of square cells whose wave vectors coincide with those of the modulational boundary pattern, we find (3.23b) and the following expression for R_2:

$$R_2 = \frac{17\pi^4}{14} - (2\sqrt{2})\pi^3 L. \tag{3.25}$$

No other solutions are possible, unless $L = 0$. The expression for R_2 given by (3.24) is less than (greater than) the expression for R_2 given by (3.25) if L is less than (greater than $l_1 = 0.259$. Hence the preferred flow pattern is that due to two-dimensional rolls along $\boldsymbol{k}_1^{(b)}$ or $\boldsymbol{k}_2^{(b)}$ for $L < l_1$, while the square pattern convection is preferred for $L > l_1$. The wave vectors for these square cells coincide with those due to boundary modulation.

Example 3. $N^{(b)} = 3$, $C_m^{(b)} = (6)^{-\frac{1}{2}}$ and $\alpha_m^{(b)} = \alpha_c$ for all m. Following the procedure discussed in the above two examples, we find the following results. For $L = 0$ or for the case where no $\boldsymbol{k}_n$ are along any $\boldsymbol{k}_m^{(b)}$, then orientationally degenerate two-dimensional rolls are stable. For the case where the wave vectors of the flow pattern coincide with a subset of the wave vectors due to the boundary modulation, we find that for L in the ranges $0 < L < l_1$, $l_1 < L < l_2$ and $l_2 < L$, two-dimensional rolls, a rectangular pattern and a hexagonal pattern are preferred, respectively. Here $l_1 = 0.432\pi$ and $l_2 = 3.06\pi$.

Example 4. $N^{(b)} = 4$, $C_m^{(b)} = 1/(2\surd 2)$ and $\alpha_m^{(b)} = \alpha_c$ for all m. Following the procedure discussed in the first two examples, we find the following results for the case where the wave vectors of the flow pattern coincide with a subset of the wave vectors of the boundary modulation. For $L < l_1$, two-dimensional rolls are preferred. For $l_1 < L < l_{2r}$, squares are preferred. For $l_{2r} < L < l_2$, rectangular (non-square) patterns are preferred. For $l_2 < L < l_3$, six-sided polygonal patterns are preferred. For $l_3 < L$, a multi-modal ($N = 4$) pattern is preferred. Here $l_i (i = 1, 2\mathrm{r}, 2, 3)$ are constants which can be determined by the procedure similar to that outlined in example 2 for the determination of l_1.

The four examples presented above indicate a general theory for arbitrary $N^{(b)}$ and for the case where the wave vectors of the flow pattern coincide with a subset of the wave vectors of the boundary modulation. Such a theory, to be discussed below, is consistent with the results for $N^{(b)} = 1, 2, 3, 4$. However, we have not been able to find a rigorous proof for arbitrary $N^{(b)}$. There exist positive constants $l_i (i = 1, 2, 2\mathrm{r}, \ldots, m; m = N^{(b)} - 1)$ such that $l_{i-1} \leqslant l_{i\mathrm{r}} \leqslant l_i < l_{i+1}$ for all i. Here a subscript r denotes l_i associated with a particular regular solution. For $L \leqslant l_1$, two-dimensional roll convection ($N = 1$) is preferred. For $l_1 < L < l_{2r}$, square pattern convection ($N = 2$) is preferred only if the wave vectors of such regular pattern are included in the expression (3.14) for $h(x, y)$. Otherwise, $l_{2r} = l_2$. For $l_{2r} < L < l_2$, rectangular pattern convection ($N = 2$) is preferred only if the wave vectors of such a non-regular pattern are included in the expression for h. Otherwise, $l_{2r} = l_2$, For $l_{j-1} < L < l_{jr} (2 \leqslant j \leqslant m)$, regular multi-modal ($N = J$) pattern convection is preferred only if the wave vectors of such a regular pattern are included in the expression for h. Otherwise, $l_{jr} = l_{j-1}$. For $l_{jr} < L < l_j$, non-regular multi-modal ($N = J$) pattern convection is preferred only if the wave vectors of such a non-regular pattern are included in the expression for h. Otherwise, $l_{jr} = l_j$. For $L > l_m$, multi-modal ($N = m + 1$) pattern convection is preferred.

4. The case of non-resonant wavelength excitation ($\gamma_n^{(b)} \equiv \gamma^{(b)} \neq \gamma_c$)

This case corresponds to the critical regime where $R \approx R_c$ and we shall show that $O(\epsilon^2) \leqslant \delta < O(\epsilon)$. Following Pal & Kelly (1978), we consider the following expressions for ϕ, θ and R in powers of ϵ and δ:

$$\begin{pmatrix} \phi \\ \theta \\ R \end{pmatrix} = \sum_{m=0} \sum_{n=0} \epsilon^m \delta^n \begin{pmatrix} \phi_{mn} \\ \theta_{mn} \\ R_{mn} \end{pmatrix}; \quad \phi_{00} = \theta_{00} = 0. \tag{4.1}$$

It can be seen from (4.1) that the terms with $m = 0$ represent the convection induced by the boundary non-uniformity, while the terms with $n = 0$ represent three-dimensional convection which can exist for a symmetric layer when $R \geqslant R_c$ and $\delta = 0$. Upon inserting (4.1) into (2.3*b*), (2.3*c*), (2.5)–(2.6) and disregarding the quadratic terms, we find the linear problem whose order-$\epsilon^1\delta^0$ system is given by a system of the form (3.2), provided that ϕ_1, θ_1 and R_0 are replaced, respectively, by ϕ_{10}, θ_{10} and R_{00}. This system has a solution of the form (3.3)–(3.8). The order-$\epsilon^0\delta^1$ system of the linear problem is of the form

$$\Delta_2(\nabla^2\phi_{01}+\theta_{01}) = 0, \tag{4.2a}$$

$$\nabla^2\theta_{01} - R_c\,\Delta_2\,\phi_{01} = 0, \tag{4.2b}$$

$$\phi_{01} = \theta_{01} - R_c\,h = 0 \quad \text{at} \quad z = -\tfrac{1}{2}, \tag{4.2c}$$

$$\phi_{01} = \theta_{01} = 0 \quad \text{at} \quad z = \tfrac{1}{2}. \tag{4.2d}$$

The general solution of (4.2) can be written as

$$\begin{pmatrix}\phi_{01}\\ \theta_{01}\end{pmatrix} = \sum_{n=-N^{(b)}}^{N^{(b)}} L\begin{pmatrix} f_n(z)\\ g_n(z)\end{pmatrix} C_n^{(b)}\,W_n^{(b)}, \tag{4.3}$$

where f_n and g_n are the solutions of the following system of equations:

$$\left.\begin{aligned} [\mathrm{D}^2-(\alpha_n^{(b)})^2]f_n + g_n &= 0,\\ [\mathrm{D}^2-(\alpha_n^{(b)})^2]\,g_n + R_c(\alpha_n^{(b)})^2 f_n &= 0,\\ f_n = g_n - R_c = 0 \quad &\text{at} \quad z = -\tfrac{1}{2},\\ f_n = g_n = 0 \quad &\text{at} \quad z = \tfrac{1}{2}, \end{aligned}\right\} \tag{4.4}$$

where $\mathrm{D} \equiv \mathrm{d}/\mathrm{d}z$. The solution to the system (4.4) is given in Appendix B. It is of interest to note that these results, (B 1)–(B 3), indicate that the double series expansion procedure of this section breaks down for $\alpha_n^{(b)} = \alpha_c$ since g_n and f_n become unbounded. Hence, our method of solution here is strictly valid for $\gamma_n^{(b)} \neq \gamma_c$.

Some preliminary investigations indicated that the trivial result that the boundary modulation controls the flow patterns will be obtained if $\delta \geqslant O(\epsilon)$. Hence non-trivial results are due to cases where $\delta < O(\epsilon)$. However, consideration of the series expansion for R given in (4.1) indicates that the present weak imperfection case can lead to non-trivial results only if $R_{01}\,\delta \gg R_{20}\,\epsilon^2$, where $R_{20} = R_{2c}$ is the classical expression for R_2 introduced in (3.19). Hence

$$O(\epsilon^2) < \delta < O(\epsilon). \tag{4.5}$$

This result implies the need for the expression for R_{01} which is found by applying the solvability condition for the order-$\epsilon\delta$ system of the nonlinear problem. It is

$$R_{01}\,\alpha_c^2\langle\theta_{01n}\,\phi_{10}\rangle = \langle\theta_{10n}(\boldsymbol{\Omega}\phi_{10}\cdot\boldsymbol{\nabla}\theta_{01} + \boldsymbol{\Omega}\phi_{01}\cdot\boldsymbol{\nabla}\theta_{10})\rangle, \tag{4.6}$$

where θ_{10n} has the same expression as θ_{1n} introduced in (3.10). Using (3.3) and (4.3), (4.6) can be simplified to the following form:

$$R_{01}\,c_n^* = \sum_{l,\,p} LS_{lp}\,c_l\,c_p^{(b)}\langle w_n\,w_l\,w_p^{(b)}\rangle, \tag{4.7}$$

where the expression for the coefficient S_{lp}, which is a function of $\alpha_p^{(b)}$ and $\phi_{lp}^{(b)}$, is given in the Appendix B and

$$\phi_{lp}^{(b)} = (\boldsymbol{k}_l\cdot\boldsymbol{k}_p^{(b)})/(\pi\alpha_p^{(b)}).$$

It can be seen from (4.7) that R_{01} can be non-zero only if

$$\boldsymbol{k}_n + \boldsymbol{k}_l + \boldsymbol{k}_p^{(b)} = 0 \tag{4.8}$$

for at least some l and p. However, for $\alpha_p^{(b)} = 2\pi$, R_{01} is zero if (4.8) is satisfied since S_{lp} given by (B 4) vanishes. Using (3.6), (3.8) and (3.15), we find that the condition (4.8) cannot be satisfied if

$$\alpha_p^{(b)} > 2\pi. \tag{4.9}$$

If the condition (4.9) is satisfied for all $p (p = -N^{(b)}, \ldots -1, 1, \ldots, N^{(b)})$, then $R_{01} = 0$ and the dominant effects can be isolated by considering the terms

$$R_{02}\,\delta^2 + R_{11}\,\epsilon\delta + R_{20}\,\epsilon^2$$

in the series expansion (4.1) for R. It is seen from the above expression that if $O(\epsilon^2) \leqslant \delta^2 \ll O(1)$, then $O(\epsilon) \leqslant \delta \ll 1$ and the trivial result that the boundary modulation removes the pattern degeneracy at the linear level follows. On the other hand, if $\delta^2 \ll O(\epsilon^2)$, the R_{20} term dominates the other terms in the above expression, implying negligible imperfection effect. Hence, non-trivial results due to significant boundary modulations exist only if

$$\alpha_p^{(b)} < 2\pi \tag{4.10}$$

for at least some p.

We shall assume that the condition (4.10) is valid. It is of interest to note that the expression for R_{01} given by (4.7) can always be negative for the boundary modulations represented either by the function h, given by (3.14), or by the function $(-h)$. For $\alpha_p^{(b)} < 2\pi$, we evaluated the expression (4.7) for R_{01} using the results given in Appendix B and found that $R_{01} < 0$ for $Lc_p^{(b)} > 0$, while $R_{01} > 0$ for $Lc_p^{(b)} < 0$. Hence subcritical instability is possible.

Let us now consider the following specific examples in order to illustrate the non-trivial and often surprising inter-relations between the boundary modulation pattern and the subsequent preferred flow pattern.

Example 1. $N^{(b)} = 1, c_m^{(b)} = 1/\sqrt{2}$, $\alpha_m^{(b)} < 2\pi$ and all the $\alpha_m^{(b)}$ have the same value. The expression on the right-hand side of (4.7) can be non-zero only if

$$\phi_{lp}^{(b)} = \phi_{np}^{(b)} = \alpha_p^{(b)}/(2\pi). \tag{4.11}$$

This result implies that the preferred solution corresponds to rectangular pattern convection where the angle w between two adjacent wave vectors is either

$$w = 2\cos^{-1}[\alpha_p^{(b)}/(2\pi)] \quad \text{or} \quad 180^\circ - w. \tag{4.12}$$

For $\alpha_p^{(b)} = \sqrt{2}\pi$, square pattern connection is preferred. Since $\alpha_p^{(b)} \neq 0$, two-dimensional roll convection is not possible.

Example 2. $N^{(b)} = 2$ (regular modulation pattern), $c_m^{(b)} = \frac{1}{2}$, $\alpha_m^{(b)} < 2\pi$ and all the $\alpha_m^{(b)}$ have the same value. The expression on the right-hand side of (4.7) can be non-zero only if (4.11) holds for either $p = 1$ or $p = 2$. This result, together with (4.7), implies that square pattern convection is preferred for $\alpha_m^{(b)} = \sqrt{2}\pi$. For other values of $\alpha_m^{(b)}$, both rectangular and multi-modal ($N = 4$) patterns correspond to the same critical R_{01}, and thus the preferred pattern is the one due to the initial condition.

The two examples presented above can be extended to arbitrary $N^{(b)}$ and for the case where the boundary pattern is regular. For the case where $\boldsymbol{k}_m^{(b)} \cdot \boldsymbol{k}_p^{(b)} = 0$ M times and for $R_{01} < 0$ multi-modal convection patterns ($N = 2, \ldots, 2M$) are the preferred patterns for $\alpha_m^{(b)} = \sqrt{2}\pi$. For $R_{01} > 0$ multi-modal ($N = 2, \ldots, 2(N^{(b)} - M)$) patterns are all possible. For other cases, multi-modal ($2 \leqslant N \leqslant 2N^{(b)}$) patterns all correspond to the same critical R_{01}, and thus the preferred pattern is one of these due to the initial condition.

5. Discussion

In formulating the present problem we have considered a horizontal layer bounded above and below by two flat plates whose mean temperatures are maintained at constant values $\bar{T}_u$ and $\bar{T}_l$, respectively. The lower plate is then given an additional spatially modulated temperature represented by the function $\delta h(x,y)$ given by (3.14). As we have shown in Appendix A, this problem is not expected to lead to different qualitative results from those for the problem where the lower boundary's location is at $z=-\frac{1}{2}+\delta h(x,y)$. Walton (1982) has shown similar results for the case of a horizontal fluid layer with slowly increasing depth. The corrugated boundary problem can incorporate the effects of roughness elements of arbitrary shape h, say on the lower boundary, and the expression (3.14) for h is still valid, provided that $N^{(b)}$ may tend to infinity and that $\alpha_m^{(b)}$ may not all have the same value. This extension of the problem can be analysed easily by dividing the boundary modulation modes into two groups of the two different types considered in §§3 and 4. Hence, the results presented in these two sections are applicable for each of these two groups of modes. Since the preferred flow pattern corresponds to the smallest value of R and (4.5) holds for the results presented in §4 then the preferred flow pattern is due to the results presented in §4 for $R_{01}<0$, while the preferred flow pattern is due to the results presented in §3 for $R_{01}>0$.

An interesting extension of the present problem, to be described in a future contribution, is to include the spatially modulated temperature on both the upper and lower boundaries. This problem was considered by Kelly & Pal (1978) for the two-dimensional case. Such an extension can generate a mean flow when a difference in phase is allowed between the variations occurring at the upper and lower boundaries (Busse 1972) and is of interest in studies of how moving thermal waves in an otherwise homogeneous fluid induce mean flows (Busse 1972; Young, Schubert & Torrance 1972).

Pal & Kelly (1978) investigated the problem of the onset of two-dimensional Rayleigh–Bénard convection with a particular one-dimensional boundary modulation in the form of a sine wave. They found that $R_{01}=0$ and $R_{02}>0$ for the case where $\alpha_n^{(b)} \neq 2\alpha_c$, while $R_{01}<0$ for $\alpha_n^{(b)}=2\alpha_c$. Their case corresponds to two-dimensional flow with the one-dimensional boundary modulation version of the present study with $Lc_n^{(b)}>0$. As we discussed in §4, we found that $R_{01}\neq 0$ for $\alpha_n^{(b)}\neq 2\alpha_c$ only for three-dimensional flow and for $\alpha_n^{(b)}<2\alpha_c$ and that $R_{01}<0$ for $Lc_n^{(b)}>0$. Hence, our results include those due to Pal & Kelly (1978) for $\alpha_n^{(b)}<2\alpha_c$. However, our result $R_{01}=0$ for $\alpha_n^{(b)}=2\alpha_c$ is in contrast to that obtained by Pal & Kelly (1978) for this case, which may be due to the particular eigenvalue relation, $R_c=(2\alpha_c)^2$, adopted for the present flow system which was found to affect the outcome of this result.

Krettenauer & Schumann (1989) investigated by direct numerical simulations the problem of Rayleigh–Bénard convection for the case where the lower boundary height varied sinusoidally in one direction only. For the subcritical flow case ($R<R_c$), where the wavelength of the surface wave was taken to be about equal to the critical wavelength ($2\pi/\alpha_c$), they found that two-dimensional rolls along the wavy surface are the form of convection, while two sets of oblique rolls evolved for the supercritical flow case ($R>R_c$), where one surface wave was allowed in the computational box. These results are similar to the present results for $N^{(b)} = 1$ derived from example 1 given in §3 and example 1 given in §4. The same authors extended their 1989 model to turbulent convection regime (Krettenauer & Schumann 1992). They found, in particular, that the motion structure persists longer over wavy terrain than over flat surfaces and that three-dimensional motions are enforced by the terrain, and the boundary modulation is more effective for longer wavelength of the surface wave. These results are consistent with the present results. It should be noted that for a realistic comparison of the results of such studies with those in applications, notably in atmospheric cases, these studies should be extended to include more general surface corrugations of the types considered in the present study and for regimes of appropriate values of the controlling parameters.

The results of the present study together with the remark made in the first paragraph of this section indicate that boundary corrugations can affect the flow patterns allowed by the nonlinear problem significantly, reduce the orientational degeneracy of the linear problem and can lead to the surprising result that the non-regular flow pattern can be preferred. Hence, our studies show for the first time that these non-regular solutions, admitted by the nonlinear system, are not just pure theoretical solutions but they can in fact be realized in practical applications if the right surface roughness shape with the right magnitude (δL) are present in such problems. Although there have been studies on the problem of convection in a horizontal layer with one-dimensional spatially periodic temperature (Kelly & Pal 1978; Krettenauer & Schumann 1989; Yoo & Kim 1991), these investigations were either for two-dimensional flow or for $N^{(b)} = 1$ only and thus could not investigate the problem of preferred flow pattern for $N^{(b)} > 1$ which is essentially a collection of an infinite number of three-dimensional flow problems. It turns out from the present results that the most surprising results presented in §3 correspond to the cases where $N^{(b)} > 1$, while the surprising results presented in §4 correspond to the cases where $N^{(b)} \geqslant 1$.

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Appendix A

For the case where the lower boundary is corrugated, we have the following lower boundary conditions:

$$\phi = \theta - \delta R h = 0 \quad \text{at} \quad z = -\tfrac{1}{2} + \delta h. \tag{A 1}$$

We apply Taylor-series expansions for the functions ϕ and θ about $z = -\frac{1}{2}$ to obtain

$$\begin{pmatrix} \phi \\ \theta \end{pmatrix} = \begin{pmatrix} \phi \\ \theta \end{pmatrix}\bigg|_{z=-\frac{1}{2}} + \delta h \begin{pmatrix} \partial\phi/\partial z \\ \partial\theta/\partial z \end{pmatrix}\bigg|_{z=-\frac{1}{2}} + \dots .$$

For $\delta = O(\epsilon^n)\,(1 < n \leqslant 2, n = 3)$ considered in this paper, we find that

$$\begin{pmatrix} \phi \\ \theta \end{pmatrix} = \begin{pmatrix} \phi \\ \theta \end{pmatrix}\bigg|_{z=-\frac{1}{2}} + O(\epsilon^{n+1}). \tag{A 2}$$

Now the qualitative results obtained in §§3 and 4 ($n = 3, 1 < n \leqslant 2$) are based on the leading-order systems of orders less than or equal to ϵ^3 and ϵ^n, $(1 < n \leqslant 2)$, respectively. Hence, terms of order ϵ^{n+1} do not affect the qualitative results obtained in this paper and using (A 1) and (A 2) leads to (2.3b) and (2.6).

Appendix B

The solution to the system (4.4) can be written in the following form:

$$f_n(z) = \begin{cases} \sum_{j=1}^{4} e_j \exp(r_j z) & \text{for} \quad \alpha_n^{(b)} \neq 2\pi, \\ \sum_{j=1}^{2} e_j \exp(r_j z) + e_3 + e_4 z & \text{for} \quad \alpha_n^{(b)} = 2\pi, \end{cases} \tag{B 1}$$

$$g_n(z) = \begin{cases} \sum_{j=1}^{4} b_j \exp(r_j z) & \text{for} \quad \alpha_n^{(b)} \neq 2\pi, \\ \sum_{j=1}^{2} b_j \exp(r_j z) + b_3 + b_4 z & \text{for} \quad \alpha_n^{(b)} = 2\pi, \end{cases} \tag{B 2}$$

where

$$\left.\begin{array}{c} r_1 = [\alpha_n^{(b)}(\alpha_n^{(b)}+2\pi)]^{\frac{1}{2}}, \quad r_2 = -r_1, \quad r_3 = [\alpha_n^{(b)}(\alpha_n^{(b)}-2\pi)]^{\frac{1}{2}}, \quad r_4 = -r_3, \\ b_1 = -\pi^2 \exp(-\tfrac{1}{2}r_1)/\sinh r_1, \quad b_2 = \pi^2 \exp(\tfrac{1}{2}r_1)/\sinh r_1, \\ b_3 = -\pi^2 \exp(-\tfrac{1}{2}r_3)/\sinh r_3 \quad \text{for} \quad \alpha_n^{(b)} \neq 2\pi, \quad \pi^2 \quad \text{for} \quad \alpha_n^{(b)} = 2\pi, \\ b_4 = \pi^2 \exp(\tfrac{1}{2}r_3)/\sinh r_3 \quad \text{for} \quad \alpha_n^{(b)} \neq 2\pi, \quad -2\pi^2 \quad \text{for} \quad \alpha_n^{(b)} = 2\pi, \\ e_j = -[r_j^2 - (\alpha_n^{(b)})^2]\, b_j/[4\pi^2(\alpha_n^{(b)})^2] \quad \text{for} \quad \alpha_n^{(b)} \neq 2\pi \quad (j = 1, \ldots, 4); \\ \text{or for} \quad \alpha_n^{(b)} = 2\pi \quad (j = 1, 2), \\ e_j = b_j/(4\pi^2) \quad \text{for} \quad \alpha_n^{(b)} = 2\pi \quad (j = 3, 4). \end{array}\right\} \tag{B 3}$$

The expression for S_{lp} introduced in (4.7) has the following form:

$$S_{lp} = b_4 \mu \left[1 - \left(\frac{\alpha_p^{(b)}}{2\pi}\right)^2\right] + 8\pi^2 \sum_j \frac{\sinh(\frac{1}{2}r_j)}{(r_j^2+4\pi^2)} \left\{ b_j \left[1 + \left(\frac{\alpha_p^{(b)}}{2\pi}\right)\phi_{lp}^{(b)}\right] - e_j \alpha_p^{(b)}[\alpha_p^{(b)} + 2\pi\phi_{lp}^{(b)}] \right\}, \tag{B 4}$$

where

$$\mu = \begin{cases} 0 & \text{for} \quad \alpha_p^{(b)} \neq 2\pi, \\ 1 & \text{for} \quad \alpha_p^{(b)} = 2\pi, \end{cases}$$

and the summation over j runs 1 to 4 for $\alpha_p^{(b)} \neq 2\pi$ and from 1 to 2 for $\alpha_p^{(b)} = 2\pi$.

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